Notes

1. For some values of $\Pi < 1$, the household technology generates *rational addiction* as suggested by Stigler and Becker (1977) and Becker and Murphy (1988).

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Prediction Formulas for Continuous Time Linear Rational Expectations Models

by Lars Peter HANSEN and Thomas J. SARGENT

In this note we derive optimal prediction formulas to be used in solving continuous time rational expectations models. In these derivations we employ Laplace transforms in a manner analogous to the use of z transforms for solving discrete time optimal prediction problems in Hansen and Sargent (1980a, Appendix A). The formulas are intended to play the same role for continuous time models that the discrete time formulas for optimal predictions of geometric distributed leads did in Hansen and Sargent (1980a).

1. Convolutions and Prediction

Let L^1 and L^2 denote the spaces of all real-valued Borel measurable functions ϕ on $\mathbf{R}$ that are absolutely integrable and square integrable, respectively. Let W denote a random measure defined on $\mathbf{R}$ with increments that are orthogonal and second-moment stationary. In other words,

$$E\left[W\{[t_2,\ t_1)\}^2\right] = t_2 - t_1 \quad \text{for} \quad t_2 > t_1\ , \tag{1.1}$$

and

$$E\left[W\{[t_4,\ t_3)\}W\{[t_2,\ t_1)\}\right] = 0 \quad \text{for} \quad t_4 > t_3 > t_2 > t_1\ . \tag{1.2}$$

Using functions in L^2 and the random measure W, we construct second-moment stationary processes as convolutions:

$$x(t) = \int_{-\infty}^{+\infty} \phi(\tau) dW(t-\tau)\ . \tag{1.3}$$

The stochastic integral in (1.3) can be interpreted as the limit point of a mean-square convergent sequence of random variables (e.g. see Rozanov 1967). Relation (1.3) gives a convenient mapping between the space L^2 of functions and the space X of stochastic processes. It turns out that inner products on these two spaces coincide. More precisely, let ϕ_1 and ϕ_2 be any two functions in L^2. An implication of (1.1) and (1.2), is

$$\int_{-\infty}^{+\infty} \phi_1(\tau)\phi_2(\tau)d\tau = E[x_1(t)\, x_2(t)] \tag{1.4}$$

where x_1 and x_2 are given by convolution (1.3) using ϕ_1 and ϕ_2 respectively.

Let L_+^2 denote the subspace of L^2 consisting of all functions that are zero on $(-\infty, 0)$ and let L_-^2 denote the subspace of all functions that are zero on $[0, \infty)$. Clearly, L_+^2 and L_-^2 are orthogonal and $L^2 = L_-^2 \oplus L_+^2$. Any ϕ in L^2 can be decomposed uniquely into the sum of two functions $\phi^+ \in L_+^2$ and $\phi^- \in L_-^2$ via:[1]

$$\begin{aligned} \phi^+(t) &\equiv \begin{cases} \phi(t) & t \geq 0 \\ 0 & t < 0 \end{cases} \\ \phi^-(t) &\equiv \begin{cases} 0 & t \geq 0 \\ \phi(t) & t < 0 \,. \end{cases} \end{aligned} \tag{1.5}$$

To formulate the prediction problems of interest, we use the random measure W to induce a family of information sets indexed by calendar time. Let $H(t)$ denote the space of random variables $x(t)$ given by (1.3) for ϕ's restricted to be in L_+^2. It follows from (1.4) that since L_+^2 is a Hilbert space, so is $H(t)$. Furthermore, the family of Hilbert spaces $\{H(t)\}$ is increasing in the sense that if $t_2 > t_1$, then $H(t_2) \supset H(t_1)$. Since $H(t)$ is constructed using the random measure W, the least squares projection operator $P[\cdot|H(t)]$ onto the space $H(t)$ is given by

$$P\Big[\int_{-\infty}^{+\infty} \phi(\tau)dW(t-\tau) \mid H(t)\Big] = \int_{-\infty}^{+\infty} \phi^+(\tau)dW(t-\tau) \,. \tag{1.6}$$

Hence the prediction process obtained by taking a process $x \in X$ constructed as a convolution of ϕ and dW and projecting it onto $H(t)$ for each t is a convolution of ϕ^+ and dW for ϕ^+ given in (1.5).

2. Transforms

One convenient way to represent functions in L^2 and characterize mapping (1.5) involves the use of transforms. For instance, Fourier transforms are valuable in characterizing the second moment properties of processes in X. For any ϕ in $L^1 \cap L^2$, the Fourier transform of ϕ is defined to be

$$\mathcal{F}t\,(\phi)(\theta) \equiv \int_{-\infty}^{+\infty} \exp(-i\theta t)\phi(t)dt \,. \tag{2.1}$$

There is a well known extension of $\mathcal{F}t$ from $L^1 \cap L^2$ to L^2. Using this extension, the spectral density function for x generated via (1.3) is just $|\mathcal{F}t\,(\phi)|^2$.

To characterize the implied second moment properties of the solutions to prediction problems of the form (1.6), we use Laplace transforms. These transforms are defined as follows. For any ϕ in L^2 and any ρ in $\mathbf{R}$ we construct a new function $\exp(-\rho t)\phi$. This new function may or may not be in L^2 depending on the value of ρ. Whenever it is in L^2, we define the Laplace transform to be:

$$\mathcal{L}p\,(\phi)(\mathbf{c}) \equiv \mathcal{F}t\,[\exp(-\rho t)\phi](\theta) \tag{2.2}$$

where $\mathbf{c} \equiv \rho + i\theta$.

The question of interest is the following. Given the Laplace transform $\mathcal{L}p(\phi)$ of a function $\phi \in L^2$, how can we compute or characterize $\mathcal{L}p(\phi^+)$ where ϕ^+ is defined in (1.5)? To answer this question, we first study Laplace transforms of functions $\phi \in L_+^2$. For any such ϕ, $\exp(-\rho t)\phi$ is also in L_+^2 as long as $\rho > 0$. Hence the Laplace transform $\mathcal{L}p(\phi)(\mathbf{c})$ is well defined on the closed right plane $\mathbf{C}_0^+$ where $\mathbf{C}_\delta^+ \equiv \{\mathbf{c} \in \mathbf{C} : \text{real}(\mathbf{c}) \geq \delta\}$. Moreover, $\mathcal{L}p(\phi)$ is analytic in the interior of $\mathbf{C}_0^+$ (relative to $\mathbf{C}$). For $\delta > 0$ and $\mathbf{c} \in \mathbf{C}_\delta^+$,

$$\begin{aligned} |\mathcal{L}p\,(\phi)(\mathbf{c})| &\leq \int_0^\infty |\phi(t)| \exp(-\delta t)dt \\ &\leq \Big[\int_0^\infty |\phi(t)|^2 dt \int_0^\infty \exp(-2\delta t)dt\Big]^{1/2} \\ &\leq \Big[\int_0^\infty |\phi(t)|^2 dt/2\delta\Big]^{1/2} \end{aligned} \tag{2.3}$$

where the second inequality is an application of the familiar Cauchy-Schwarz Inequality. The right side of (2.3) gives a uniform bound (in

$\mathbf{c}$) on $\mathcal{L}p(\phi)$ over the set $\mathbf{C}_\delta^+$. This bound becomes arbitrarily small as δ tends to plus infinity.

Consider next functions $\phi \in L_-^2$. For any such ϕ, $\mathcal{L}p(\phi)(\mathbf{c})$ is always well defined for $\mathbf{c}$ in left half plane $\mathbf{C}_0^- \equiv \{\mathbf{c} \in \mathbf{C} : \text{real}(\mathbf{c}) \leq 0\}$, and $\mathcal{L}p(\phi)$ is analytic in the interior of that domain. Define $\mathbf{C}_\delta^- \equiv \{\mathbf{c} \in \mathbf{C} : \text{real}(\mathbf{c}) \leq \delta\}$. Mimicking the previous argument, it can be shown that for any $\delta < 0$, $\mathcal{L}p(\phi)$ is bounded on the domain $\mathbf{C}_\delta^-$ and that the bound can be made arbitrarily small by driving δ towards minus infinity.

For general functions ϕ in L^2, $\mathcal{L}p(\phi)$ may only be defined on the imaginary axis, i.e. for real$(\mathbf{c}) = 0$. We are interested in a smaller class of functions, however. Let Φ be the set of all functions $\phi \in L^2$ such that $\mathcal{L}p(\phi^-)$ is analytic in the interior of a region $\mathbf{C}_\delta^-$ for some $\delta > 0$. In this case $\mathcal{L}p(\phi)$ is analytic in the interior of the strip $\mathbf{C}_\delta^- \cap \mathbf{C}_0^+$. Furthermore, for any closed interval $J \subset (0, \delta)$, $\mathcal{L}p(\phi)$ is bounded on $\{\mathbf{c} \in \mathbf{C} : \text{real}(\mathbf{c}) \in J\}$. Define $\mathcal{A}$ to be the collection of all Laplace transforms of functions $\phi \in \Phi$.

The following result gives the decomposition for $a \in \mathcal{A}$ corresponding to the decomposition $\phi = \phi^+ + \phi^-$.

Lemma: For any $a \in \mathcal{A}$ there is a unique decomposition $a = a^+ + a^-$ where

(i) a^- is analytic in the interior of $\mathbf{C}_\delta^-$, uniformly bounded on any closed half plane $\mathbf{C}_\rho^-$ for $\rho < \delta$ and

$$\lim_{\rho \to -\infty} \max_{\mathbf{c} \in \mathbf{C}_\rho^-} |a^-(\mathbf{c})| = 0 \ ;$$

(ii) a^+ is analytic in the interior of $\mathbf{C}_0^+$, uniformly bounded on any closed half plane $\mathbf{C}_\rho^+$ for any $\rho > 0$.

Proof: Functions a^- and a^+ satisfying (i) and (ii) are obtained by letting $a^- = \mathcal{L}p(\phi^-)$ and $a^+ = \mathcal{L}p(\phi^+)$. To show that the decomposition is unique, we let $a = b^+ + b^-$ be any other decomposition where b^- satisfies (i) and b^+ satisfies (ii). Note that

$$a^+ - b^+ = b^- - a^-$$

at least in the interior of the strip $\mathbf{C}_\delta^- \cap \mathbf{C}_0^+$. Since $a^+ - b^+$ is analytic in the interior of $\mathbf{C}_0^+$ and $b^- - a^-$ is analytic in the interior of $\mathbf{C}_\delta^-$, $b^- - a^-$ can be extended to be analytic on all of $\mathbf{C}$. Furthermore, the uniform bounds on $a^+ - b^+$ and $b^- - a^-$ on overlapping half planes ensure that the extension of $b^- - a^-$ is bounded as well. The only functions that

are bounded and analytic on $\mathbf{C}$ are constant. Since a^- and b^- satisfy (i) and the extension of $b^- - a^-$ to C is constant, $a^+ - b^+$ must be identically zero. ∎

Decompositions like that given in the Lemma apply to a much more general collection of analytic functions than the Laplace transforms of functions in L^2. For instance, they also apply to Laplace transforms of generalized functions (e.g. see Beltrami and Wohlers 1966). However, these more general decompositions may not be unique. For instance, suppose we ignore the requirement

$$\lim_{\rho \to -\infty} \max_{\mathbf{c} \in \mathbf{C}_\rho^-} |a^-(\mathbf{c})| = 0$$

in (i) of the Lemma. Then one can always add complex numbers to a^+ and subtract the same numbers from a to obtain other decompositions of a. If in addition, we ignore the bound restrictions in (i) and (ii) of the Lemma then one can add functions, such as polynomials, that are analytic in the entire complex plane to a^+ and subtract them from a^- to obtain other decompositions of a. Therefore in applying the Lemma to compute $\mathcal{L}p(\phi^+)$, it is important to check whether the candidates for $\mathcal{L}p(\phi^+)$ and $\mathcal{L}p(\phi^-)$ satisfy the bounds restrictions in (i) and (ii).

3. Examples

We now apply the Lemma to obtain frequency domain characterizations of the solutions to prediction problems that occur in rational expectations models. These problems all have the following structure. Let $\psi \in L_+^2$, and define $y(t)$ by the convolution:

$$y(t) = \int_0^{+\infty} \psi(\tau) dW(t - \tau) \ .$$

Construct a new process by forming a *forward-looking* convolution using a function $\gamma \in L_-^1$: [2]

$$(3.1) \qquad x(t) = \int_{-\infty}^{+\infty} \gamma(\tau) y(t - \tau) d\tau \equiv \int_{-\infty}^{+\infty} \phi(\tau) dW(t - \tau)$$

where ϕ is given by the convolution:

$$\phi(\tau) = \int_{-\infty}^{+\infty} \gamma(s) \psi(\tau - s) ds \ .$$

Applying the well known product representation for Fourier transforms of convolutions, we have that

$$\mathcal{F}t(\phi) = \mathcal{F}t(\psi)\mathcal{F}t(\gamma) \ .$$

This same result extends to Laplace transforms on the common domain of $\mathcal{L}p(\phi)$ and $\mathcal{L}p(\gamma)$. For the examples we consider, there will exist a $\delta > 0$ such that $\mathcal{L}p(\gamma)$ is defined on the interior of $\mathbf{C}_\delta^-$. Hence on the interior of the strip $\mathbf{C}_\delta^- \cap \mathbf{C}_0^+$,

$$\mathcal{L}p(\phi) = \mathcal{L}p(\psi)\mathcal{L}p(\gamma) \; . \tag{3.2}$$

We now investigate three related examples.

Example 1:

Suppose that γ is given by

$$\gamma(t) = \begin{cases} 0 & t \geq 0 \\ \exp(\delta t) & t < 0 \end{cases} \; . \tag{3.3}$$

Then

$$\mathcal{L}p(\gamma)(\mathbf{c}) = \int_{-\infty}^{0} \exp[(\delta - \mathbf{c})t]dt = 1/(\mathbf{c} - \delta)$$

for real$(\mathbf{c}) < \delta$. Thus

$$\mathcal{L}p(\phi)(\mathbf{c}) = \mathcal{L}p(\psi)(\mathbf{c})/(\mathbf{c} - \delta) \; .$$

Note that $\mathcal{L}p(\phi)$ is analytic on $\mathbf{C}$ except possibly at the point δ where it may have a pole. If $\mathcal{L}p(\psi)(\delta)$ is zero, the singularity at δ is removable and $\mathcal{L}p(\phi^+) = \mathcal{L}p(\phi)$. Usually $\mathcal{L}p(\phi)$ will have a pole at δ, and to compute $\mathcal{L}p(\phi^+)$ we must eliminate this pole. One candidate for $\mathcal{L}p(\phi^+)$ is

$$a^+(\mathbf{c}) = [\mathcal{L}p(\psi)(\mathbf{c}) - \mathcal{L}p(\psi)(\delta)]/(\mathbf{c} - \delta) \; .$$

Notice that the singularity of a^+ at δ is removable. The corresponding choice of a^- is

$$a^-(\mathbf{c}) = a(\mathbf{c}) - a^+(\mathbf{c}) = \mathcal{L}p(\psi)(\delta)/(\mathbf{c} - \delta) \; .$$

It is straightforward to show that a^+ and a^- satisfy the requirements of the Lemma. Therefore,

$$\mathcal{L}p(\phi^+) = [\mathcal{L}p(\psi)(\mathbf{c}) - \mathcal{L}p(\psi)(\delta)]/(\mathbf{c} - \delta) \; . \tag{3.4}$$

Formula (3.4) is the continuous time counterpart to formula (5) in Hansen and Sargent (1980a).

Example 2

More generally, suppose

$$\mathcal{L}p(\gamma)(\mathbf{c}) = p_n(\mathbf{c})/p_d(\mathbf{c})$$

where p_n and p_d are finite-order polynomials with real coefficients. To ensure that $p_n(\mathbf{c})/p_d(\mathbf{c})$ is the Laplace transform of a function in L_-^2, we assume that the order of p_d exceeds the order of p_n and that the zeros of p_d are in the interior of $\mathbf{C}_0^+$. In this case

$$\mathcal{L}p(\phi)(\mathbf{c}) = \mathcal{L}p(\psi)(\mathbf{c})p_n(\mathbf{c})/p_d(\mathbf{c}) \; ,$$

which has poles in the interior of $\mathbf{C}_0^+$ only at the zeros of p_d. Let a_j denote the principal part of the Laurent series expansion of $\mathcal{L}p(\phi)(\mathbf{c})$ at the j^{th} zero of p_d. It follows from the partial fractions decomposition of a meromorphic function that

$$a^+ = \mathcal{L}p(\phi) - \sum_j a_j$$

is analytic in the interior of $\mathbf{C}_0^+$. Furthermore, the principal parts, a_j, are each sums of reciprocals of first and higher-order polynomials and hence satisfy

$$\lim_{\rho \to -\infty} \max_{\mathbf{c} \in \mathbf{C}_\rho^-} |a_j(\mathbf{c})| = 0 \; .$$

for each j. By construction,

$$a^- = \sum_j a_j$$

satisfies (i) of the Lemma where δ is the real part of the zero of p_n closest to the imaginary axis and is bounded on $\mathbf{C}_\rho^-$ for any $\rho < \delta$. Therefore, we have the following generalization of (3.4):

$$\mathcal{L}p(\phi^+) = \mathcal{L}p(\phi) - \sum_j a_j \; .$$

Example 3

Suppose that γ is given by (3.2), and $\mathcal{L}p(\psi)$ is a rational function:

$$\mathcal{L}p(\psi) = q_n/q_d$$

where q_n and q_d are polynomials with real coefficients. To guarantee that q_n/q_d is the Laplace transform of a function in L_+^2, we assume that

the order of q_d exceeds the order of q_n and that the zeros of q_d are in the interior of $\mathbf{C}_0^-$. Solution (3.4) now becomes

$$\mathcal{L}p(\phi)(\mathbf{c}) = q_n(\mathbf{c})/[q_d(\mathbf{c})(\mathbf{c}-\delta)] .$$

From Example 1, we know that

$$\begin{aligned}(3.5)\qquad \mathcal{L}p(\phi^+)(\mathbf{c}) &= [q_n(\mathbf{c})/q_d(\mathbf{c}) - q_n(\delta)/q_d(\delta)]/(\mathbf{c}-\delta) \\ &= [q_n(\mathbf{c})q_d(\delta) - q_n(\delta)q_d(\mathbf{c})]/[q_d(\mathbf{c})q_d(\delta)(\mathbf{c}-\delta)] .\end{aligned}$$

The right side of (3.5) has a removable singularity at δ by construction. This is evident because the polynomial $[q_n(\mathbf{c})q_d(\delta) - q_n(\delta)q_d(\mathbf{c})]$ has a zero at δ. Canceling the common factor $(\mathbf{c}-\delta)$ in the numerator and denominator results in

$$\mathcal{L}p(\phi^+)(\mathbf{c}) = q_n^+(\mathbf{c})/q_d^+(\mathbf{c})$$

where

$$q_d^+(\mathbf{c}) = q_d(\mathbf{c})q_d(\delta)$$

and q_n^+ satisfies

$$(3.6)\qquad q_n^+(\mathbf{c})(\mathbf{c}-\delta) = [q_n(\mathbf{c})q_d(\delta) - q_n(\delta)q_d(\mathbf{c})] .$$

By equating coefficients of the polynomials on both sides of (3.6), one can construct a linear system of equations in the coefficients of $q_n^+(\mathbf{c})$. In fact there is a recursive structure to this equation system that can be exploited as follows. Let η_j denote the coefficient on $\mathbf{c}^j$ in $[q_n(\mathbf{c})q_d(\delta) - q_n(\delta)q_d(\mathbf{c})]$ and let ϵ_j denote the corresponding coefficient in $q_n^+(\mathbf{c})$. Then

$$-\delta\epsilon_0 = \eta_0$$

and

$$\epsilon_{j-1} - \delta\epsilon_j = \eta_j \quad \text{for } j \geq 1$$

which can be solved recursively beginning with ϵ_0. The solution to this recursion gives a continuous time counterpart to formulas reported in Hansen and Sargent (1980a, 1981b) for autoregressive and autoregressive moving-average processes.

4. Vector Information Structures

Suppose that W is an k-dimensional vector random measure with second moment stationary increments. We now replace (1.1) and (1.2) with

$$E\left[W\{[t_2, t_1)\}W\{[t_2, t_1)\}'\right] = (t_2 - t_1)I_k \quad \text{for } t_2 > t_1 ,$$

and

$$E\left[W\{[t_4, t_3)\}W\{[t_2, t_1)\}'\right] = 0 \quad \text{for } t_4 > t_3 > t_2 > t_1$$

where I_k is a k-dimensional identity matrix. Processes in X are now constructed using a k-dimensional vector ϕ of functions in L^2 via:

$$x(t) = \int_{-\infty}^{+\infty} \phi(t) \cdot dW(t-\tau) .$$

The analyses in Sections 2 and 3 extend by applying the decompositions to each of the k Laplace transforms of entries in ϕ. In Example 1 formula (3.4) still applies where $\mathcal{L}p(\psi)$ is the vector of Laplace transforms of entries in ψ. The recursions derived in Example 3 still apply where q_n is now a k-dimensional vector of polynomials, each with orders less than the scalar polynomial q_d.

5. Nonstationarities

In Section 3, the assumption that $\psi \in L_+^2$ guaranteed that process y is second moment stationary. Our analysis can be extended to a more general class of processes, however. To accommodate nonstationarities, it is most convenient to think of the underlying information process as starting at some initial time, say $t = 0$. Hence we imagine (1.1) and (1.2) holding for nonnegative values of t_1, t_2, t_3 and t_4, and we assume that the random measure of any interval contained in $(-\infty, 0)$ is zero. This permits formula (3.1) to be well defined for a much larger class of functions ψ. We might view the process y as being the deviation from a path that is perfectly predictable from time zero forward. We impose the weaker requirement that $\exp(-\rho t)\psi$ be in L^2 for strictly positive values of ρ which allows for polynomial growth in the second moment of y.[3] The calculations in Examples 1 through 3 still apply. In the case of Example 3, to accommodate polynomial growth we now allow q_d to have zeros on the imaginary axis of the complex plane $\mathbf{C}$.

Notes

1. The uniqueness of this decomposition requires some qualification. Elements of L^2 are only defined up to an equivalence of functions that are equal almost everywhere. Hence from the vantage point of L^2, the construction of ϕ^+ and ϕ^- at a particular point, say $t = 0$, is inconsequential.

2. We take the right side of equation (3.1) as the definition of $x(t)$. Alternatively, for particular classes of γ we could define $x(t)$ using finite sum approximations for the middle integral.

3. In Examples 1 and 3 it is also possible to allow for exponential growth in the second moments of y as long as $\psi \exp(-\sigma t)$ is in L^2_+ for some σ satisfying $0 < \sigma < \delta$. The transform analysis now applies to the narrower strip $\mathbf{C}^-_\delta \cap \mathbf{C}^+_\sigma$.

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Identification of Continuous Time Rational Expectations Models from Discrete Time Data

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1. Introduction

This paper proves two propositions about identification in a continuous time version of a linear stochastic rational expectations model. The model is a continuous time version of Lucas and Prescott (1971), in which the equilibrium can be interpreted as the solution of a stochastic control problem, either of a collection of private agents or of a fictitious *social planner*. Estimation is directed toward isolating the parameters of the *agent's* objective function and of the stochastic processes of the forcing functions that the agent faces. This approach has been advocated by Lucas (1967, 1976), Lucas and Prescott (1971), and Lucas and Sargent (1981) as offering the potential to analyze an interesting class of policy interventions promised by *structural* models, while meeting the criticisms of most econometric policy evaluation methods that were made by Lucas (1976). At the same time, inspired by the work of Sims (1971), Geweke (1978), and P.C.B. Phillips (1972, 1973, 1974), we want to estimate models in which optimizing economic agents make decisions at finer time intervals than the interval of time between the observations used by the econometrician. We adopt a continuous time theoretical framework both because it is an interesting limiting case, and because it has received extensive attention in the theoretical and the econometric literatures.

Identification of the parameters of a continuous time model from discrete time data must confront the aliasing problem (see, e.g., Phillips 1973). In general, there is an uncountable infinity of continuous time models that are consistent with a collection of discrete time observations. However, with finite parameter continuous time models, the